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An Inequality for Tchebycheff Polynomials and Extensions

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The inequality $T_n(xy) \leqslant T_n(x)\,T_n(y)$, $x, y \geqslant 1$, where $T_n(x)$ is the Tchebycheff polynomial of the first kind, can be proven very easily by use of one of the extremal properties of these polynomials. It also follows from $(d^2/du^2)\log T_n(e^u) \leqslant 0$, $u \geqslant 0$. Various proofs are given for these inequalities and for generalizations to other classes of polynomials.

INTRODUCTION

One of the important properties of the Tchebycheff polynomials is an extremal property outside $(-1, 1)$: if $p_n(x)$ is an arbitrary polynomial of degree n which satisfies

$$|p_n(x)| \leqslant 1, \qquad -1 \leqslant x \leqslant 1, \tag{0.1}$$

and $T_n(x)$ is the Tchebycheff polynomial defined by $T_n(\cos\theta) = \cos n\theta$, then

$$|p_n(x)| \leqslant T_n(x), \qquad x \geqslant 1. \tag{0.2}$$

For this and several other properties of polynomials see Rogosinki [9] and Schur [10].

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The main results of the present paper stem from the observation that this property of Tchebycheff polynomials gives

$$T_n(xy) \leqslant T_n(x)\, T_n(y), \qquad x, y \geqslant 1, \tag{0.3}$$

and that this inequality also follows from

$$\frac{d^2}{du^2} \log T_n(e^u) \leqslant 0, \qquad u \geqslant 0. \tag{0.4}$$

Various proofs are given for these inequalities and for generalizations to other classes of polynomials. For instance, it is shown that the ultraspherical polynomials $C_n^\lambda(x)$ satisfy

$$\begin{aligned}\frac{C_n^\lambda(xy)}{C_n^\lambda(1)} &\leqslant \frac{C_n^\lambda(x)}{C_n^\lambda(1)}\frac{C_n^\lambda(y)}{C_n^\lambda(1)} \\ &\leqslant \frac{C_n^\lambda(xy + (x^2-1)^{1/2}(y^2-1)^{1/2})}{2C_n^\lambda(1)} \\ &\quad + \frac{C_n^\lambda(xy - (x^2-1)^{1/2}(y^2-1)^{1/2})}{2C_n^\lambda(1)}\end{aligned} \tag{0.5}$$

for $x, y \geqslant 1$, $\lambda > 0$.

1. Proof of (0.3) and Some Extensions

Fix $y \geqslant 1$ and consider

$$p_n(x) = T_n(xy)/T_n(y)$$

for $|x| \leqslant 1$. Clearly, $|T_n(u)| \leqslant 1$ when $|u| \leqslant 1$ and $|T_n(u)|$ is an increasing function of $|u|$ when $|u| \geqslant 1$. Thus $p_n(x)$, which is a polynomial of degree n, satisfies (0.1); so (0.2) gives

$$|T_n(xy)/T_n(y)| \leqslant T_n(x), \qquad x, y \geqslant 1,$$

which then gives (0.3) since $T_n(x) \geqslant 1$ for $x \geqslant 1$.

To extend (0.3) to other polynomials it is useful to generalize it to

$$T_n(r)\, T_n(s) \leqslant T_n(x)\, T_n(y), \qquad 1 \leqslant r \leqslant x \leqslant y \leqslant s, \qquad rs = xy, \tag{1.1}$$

which is equivalent to

$$\frac{d^2}{du^2} \log T_n(e^u) \leqslant 0, \qquad u \geqslant 0. \tag{1.2}$$

This leads us to the following general result.

THEOREM 1. *Let* $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *be a polynomial with only real roots and suppose that* $a_n > 0$ *and* $a_{n-1} \leqslant 0$. *Let* c *be an upper bound for the roots of* $p_n(x)$. *Then*

$$p_n(r)\, p_n(s) \leqslant p_n(x)\, p_n(y) \tag{1.3}$$

whenever $c \leqslant r \leqslant x \leqslant y \leqslant s$ *and* $rs = xy$. *In particular, if* $c = 1$ *and* $p_n(1) \geqslant 1$, *then*

$$p_n(xy) \leqslant p_n(x)\, p_n(y), \qquad x, y \geqslant 1. \tag{1.4}$$

Note that the condition "$a_n > 0$" can be written "$p_n(x) \to \infty$ as $x \to \infty$" and when this holds the condition "$a_{n-1} \leqslant 0$" can be written "$\sum_{k=1}^{n} x_k \geqslant 0$, where $x_1, \ldots, x_n$ are the roots of $p_n(x)$." Theorem 1 applies to most of the classical orthogonal polynomials [11].

COROLLARY 1. *Inequality* (1.3) *holds in each of the following cases:*

(a) $p_n(x)$ *is a Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ *with* $\beta \geqslant \alpha > -1$ *and* $c = 1$,

(b) $p_n(x)$ *is a generalized Laguerre polynomial* $L_n^{\alpha}(x)$ *with* $\alpha > -1$ *and* $c = 2n + \alpha + 1 + \{(2n + \alpha + 1)^2 - \alpha^2 + 1/4\}^{1/2}$,

(c) $p_n(x)$ *is the Hermite polynomial* $H_n(x)$ *and* $c = (2n + 1)^{1/2}$.

In particular, if $p_n(x)$ *is the Legendre polynomial* $P_n(x)$ *or the Tchebycheff polynomial* $T_n(x)$, *then*

$$p_n(xy) \leqslant p_n(x)\, p_n(y), \qquad x, y \geqslant 1.$$

Proof of Theorem 1. Our hypotheses imply that the function

$$g(u) = \log p_n(e^u)$$

is defined for all real u with $e^u > c$. Since (1.3) is equivalent to

$$g''(u) \leqslant 0 \text{ when } e^u > c, \tag{1.5}$$

it suffices to prove (1.5). Let $x_1, \ldots, x_n$ be the roots of $p_n(x)$. Then

$$g''(u) = \frac{d}{du}\left[e^u \frac{p_n'(e^u)}{p_n(e^u)}\right] = \frac{d}{du}\, e^u \sum_{k=1}^{n} \frac{1}{e^u - x_k}$$

$$= \frac{d}{du} \sum_{k=1}^{n} \left(1 + \frac{x_k}{e^u - x_k}\right) = -e^u \sum_{k=1}^{n} \frac{x_k}{(e^u - x_k)^2}\,.$$

Partition the numbers 1,..., n into two disjoint sets J and K so that $x_j \geqslant 0$ for $j \in J$ and $x_k < 0$ for $k \in K$. Then

$$\sum_{k=1}^{n} \frac{x_k}{(x-x_k)^2} = \sum_{j\in J} \frac{x_j}{(x-x_j)^2} - \sum_{k\in K} \frac{|x_k|}{(x+|x_k|)^2}$$

$$\geqslant \frac{1}{x^2}\Big(\sum_{j\in J} x_j - \sum_{k\in K} |x_k|\Big) = \frac{1}{x^2}\sum_{k=1}^{n} x_k$$

$$= -\frac{a_{n-1}}{a_n x^2} \geqslant 0,$$

for $x > c$. Therefore $g''(u) \leqslant 0$ when $e^u > c$, as desired.

Proof of Corollary 1. To prove that (1.3) holds for the case (a), we substitute $P_n^{(\alpha,\beta)}(x)$ into its associated homogeneous linear differential equation [11, p. 60] and collect the coefficients of x^{n-1} to obtain

$$(2n+\alpha+\beta)\, a_{n-1} + (\beta-\alpha)\, na_n = 0.$$

Since $a_n > 0$, it follows that $a_{n-1} \leqslant 0$. This can also be shown by using an explicit formula for Jacobi polynomials [11, (4.21.2)]. We may take $c = 1$ since the roots of $P_n^{(\alpha,\beta)}(x)$ lie in the interval $(-1, 1)$. The proofs for cases (b) and (c) are similar. (In (b) we used the fact that if (1.3) holds for some $p_n(x)$, then it also holds with $p_n(x)$ replaced by $(-1)^n p_n(x)$.) Bounds for the roots of $p_n(x)$ in these two cases follow from [11, (6.31.7) and (6.32.3)]. The last assertion holds since $P_n(x)$ and $T_n(x)$ are included under case (a) and

$$P_n(1) = T_n(1) = 1.$$

Note that, by the Gauss theorem on the zeros of polynomial derivatives, the conclusions of Theorem 1 also hold for all derivatives of $p_n(x)$. A different type of extension of (0.3) to derivatives of Tchebycheff polynomials is given in the following theorem.

THEOREM 2. *Let j and k be nonnegative integers with $j + k \leqslant n$. Then*

$$y^k T_n^{(j+k)}(xy) \leqslant T_{n-j}^{(k)}(x)\, T_n^{(j)}(y) \tag{1.6}$$

for $x, y \geqslant 1$, where $T_n^{(k)}(x) = (d^k/dx^k)\, T_n(x)$.

Proof. Let $y \geqslant 1$ and put $p(x) = T_n^{(j)}(xy)/T_n^{(j)}(y)$. Then $p(x)$ is a polynomial of degree $n - j$. Moreover,

$$|p(x)| \leqslant 1, \qquad -1 \leqslant x \leqslant 1. \tag{1.7}$$

To see this, first note that an easy induction on j shows that

$$T_n^{(j)}(t) = \sum_{i=1}^{n-j} A_{ij} T_i(t),$$

where $A_{ij} \geqslant 0$. Hence since each $T_i(t)$ assumes its maximum absolute value on the interval $[-y, y]$ at the point $t = y$, the same is true of $T_n^{(j)}(t)$. From (1.7) and a well-known extension of the extremal property for Tchebycheff polynomials (see [9] or [10]), it follows from (1.7) that

$$| p^{(k)}(x)| \leqslant T_{n-j}^{(k)}(x)$$

for $x \geqslant 1$, which is (1.6). (The proof of (1.7) given here was suggested by T. J. Rivlin. Another proof can be obtained from [11, (4.21.7) and (7.32.2)].)

From the case $x = y = 1$ of (1.6), which is not entirely obvious, it is clear that Theorem 2 is weaker than the following:

THEOREM 3. *Let j and k be nonnegative integers with $j + k \leqslant n$. Then*

$$y^k \frac{T_n^{(j+k)}(xy)}{T_n^{(j+k)}(1)} \leqslant \frac{T_{n-j}^{(k)}(x)}{T_{n-j}^{(k)}(1)} \frac{T_n^{(j)}(y)}{T_n^{(j)}(1)}, \qquad x, y \geqslant 1. \tag{1.8}$$

Proof. Let $c_n(x; \lambda) = C_n^{\lambda}(x)/C_n^{\lambda}(1)$, $\lambda > -1/2$, where $C_n^{\lambda}(x)$ is the ultraspherical polynomial [3, p. 174]. Then $c_{n-k}(x; k) = T_n^{(k)}(x)/T_n^{(k)}(1)$, and so (1.8) is equivalent to

$$y^k c_{n-j-k}(xy; j + k) \leqslant c_{n-j-k}(x; k)\, c_{n-j}(y; j), \quad x, y \geqslant 1. \tag{1.9}$$

To prove (1.9) first use part (a) of Corollary 1 (which applies since

$$c_n(x; \lambda) = P_n^{(\alpha,\alpha)}(x)/P_n^{(\alpha,\alpha)}(1)$$

with $\alpha = \lambda - \frac{1}{2}$) to obtain

$$y^k c_{n-j-k}(xy; j + k) \leqslant y^k c_{n-j-k}(x; j + k)\, c_{n-j-k}(y; j + k)$$

for $x, y \geqslant 1$. Next use the inequality (proved below)

$$c_n(x; \lambda) \leqslant c_n(x; \mu), \qquad x \geqslant 1, \qquad \lambda > \mu > -\tfrac{1}{2}, \tag{1.10}$$

on the three factors:

$$c_{n-j-k}(x; j + k) \leqslant c_{n-j-k}(x; k),$$
$$c_{n-j-k}(y; j + k) \leqslant c_{n-j-k}(y; j),$$
$$y^k \leqslant c_k(y; j), \qquad \text{(recall that } y^k = \lim_{\lambda\to\infty} c_k(y; \lambda))$$

to obtain

$$y^k c_{n-j-k}(xy; j+k) \leqslant c_k(y; j)\, c_{n-j-k}(x; k)\, c_{n-j-k}(y; j), \qquad x, y \geqslant 1.$$

But

$$c_k(y; j)\, c_{n-j-k}(y; j) \leqslant c_{n-j}(y; j), \qquad y \geqslant 1, \tag{1.11}$$

(proved below), so (1.9) holds.

Thus there remains only the problem of proving (1.10) and (1.11). Consider (1.11) first. For $\lambda \geqslant 0$ it is known (see [11, p. 390, Exercise 84] or [4]) that

$$c_n(x; \lambda)\, c_m(x; \lambda) = \sum_{k=|n-m|}^{n+m} A(k, m, n)\, c_k(x; \lambda),$$

with $A(k, m, n) \geqslant 0$ and $\sum_k |A(k, m, n)| = 1$. If we can show that

$$c_k(x; \lambda) \leqslant c_{n+m}(x; \lambda), \qquad x \geqslant 1, \quad k \leqslant n+m, \tag{1.12}$$

then (1.11) clearly holds. We can show that (1.12) holds for an even wider class of orthogonal polynomials. Let $p_n(x)$ be a set of polynomials orthogonal on $(-1, 1)$ with respect to a positive measure on $(-1, 1)$ and assume

$$p_n(-x) = (-1)^n p_n(x)$$

(i.e., the measure is even) and $p_n(1) = 1$. Then

$$xp_n(x) = a_n p_{n+1}(x) + (1 - a_n)\, p_{n-1}(x), \quad p_0(x) = 1, \quad p_1(x) = x,$$

with $0 < a_n < 1$. Conversely this recurrence formula implies that $p_n(x)$ is orthogonal on $(-1, 1)$ with respect to a positive even measure. Then

$$\begin{aligned} a_n[p_{n+1}(x) - p_n(x)] &= xp_n(x) - a_n p_n(x) - (1 - a_n)\, p_{n-1}(x) \\ &\geqslant (1 - a_n)\, p_n(x) - (1 - a_n)\, p_{n-1}(x) \\ &= (1 - a_n)[p_n(x) - p_{n-1}(x)] \\ &\geqslant \cdots \geqslant K_n[p_1(x) - p_0(x)] \geqslant 0, \end{aligned}$$

for $x \geqslant 1$, where $K_n > 0$. This gives (1.12).

Thus there remains only (1.10). Recall Gegenbauer's formula (see [6] or [1])

$$c_n(x; \lambda) = \sum_{k=0}^{n} B(k, n)\, c_k(x; \mu), \qquad B(k, n) \geqslant 0, \quad \lambda > \mu > -\tfrac{1}{2}. \tag{1.13}$$

Then $\sum_k B(k, n) = 1$, and so

$$c_n(x; \lambda) = \sum_{k=0}^{n} B(k, n)\, c_k(x; \mu) \leqslant \sum_{k=0}^{n} B(k, n)\, c_n(x; \mu) = c_n(x; \mu)$$

for $x \geqslant 1$, $\lambda > \mu > -\frac{1}{2}$, which completes the proof.

2. Concavity of $\log | p_n(e^u)|$

Since the restriction $x, y \geqslant 1$ in (0.3) cannot be relaxed to $x, y \geqslant c$ with $c < 1$, it is of interest to note that (0.4) extends to

$$\frac{d^2}{du^2} \log | T_n(e^u)| \leqslant 0, \qquad -\infty < u < \infty, \qquad T_n(e^u) \neq 0.$$

This is a special case of

THEOREM 4. *Let* $p_n(x) = a_n \prod_{k=1}^{n}(x - x_k)$ *with* $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ *and* $x_{n+1-k} = -x_k$, $k = 1, 2,..., n$. *If* $-\infty < u < \infty$ *and* $e^u \neq x_k$ *for any* k, *then*

$$\frac{d^2}{du^2} \log | p_n(\pm e^u)| \leqslant 0, \tag{2.1}$$

with equality if and only if each $x_k = 0$.

Proof. Let $g(u) = \log | p_n(e^u)|$ and $x = e^u$. Then proceeding as in the proof of Theorem 1, we have

$$\begin{aligned} g''(u) &= -e^u \sum_{k=1}^{n} \frac{x_k}{(e^u - x_k)^2} \\ &= -x \sum_{k=1}^{[(n+1)/2]} x_k \left(\frac{1}{(x - x_k)^2} - \frac{1}{(x + x_k)^2}\right) \\ &= -4x^2 \sum_{k=1}^{[(n+1)/2]} \frac{x_k^{\,2}}{(x^2 - x_k^{\,2})^2}, \end{aligned}$$

which gives (2.1) for $p_n(+e^u)$. The result for $p_n(-e^u)$ then follows from $| p_n(x)| = | p_n(-x)|$.

For polynomials with only nonnegative zeros we have the following logarithmic concavity and convexity results.

THEOREM 5. *Let* $p_n(x) = a_n \prod_{k=1}^{n}(x - x_k)$ *with* $x_k \geqslant 0$, $k = 1,..., n$. *If* $-\infty < u < \infty$ *and* $e^u \neq x_k$ *for any* k, *then*

$$\frac{d^2}{du^2} \log | p_n(e^u)| \leqslant 0, \qquad \frac{d^2}{du^2} \log | p_n(-e^u)| \geqslant 0, \tag{2.2}$$

with equality if and only if each $x_k = 0$.

Proof. Follows directly from the identities

$$\frac{d^2}{du^2} \log | p_n(e^u)| = -e^u \sum_{k=1}^{n} \frac{x_k}{(e^u - x_k)^2},$$

$$\frac{d^2}{du^2} \log | p_n(-e^u)| = e^u \sum_{k=1}^{n} \frac{x_k}{(e^u + x_k)^2}.$$

Note that if $p_n(x)$ has only negative roots then the inequalities in (2.2) must be reversed. In particular, since the root of

$$P_1^{(\alpha,\beta)}(x) = [(\alpha + \beta + 2)x + \alpha - \beta]/2$$

is negative when $\alpha > \beta > -1$, we find that $(d^2/du^2) \log P_1^{(\alpha,\beta)}(e^u) > 0$, $u \geqslant 0$, $\alpha > \beta > 1$; from which it follows that the restriction $\beta \geqslant \alpha$ in part (a) of Corollary 1 cannot be relaxed. However, since all of the zeros of $P_n^{(\alpha,\beta)}(2x - 1)$ lie in the interval $(0, 1)$ when $\alpha, \beta > -1$, from Theorem 5 we have the following inequality.

COROLLARY 2. *If* $\alpha, \beta > -1$, then

$$P_n^{(\alpha,\beta)}(2r - 1)\, P_n^{(\alpha,\beta)}(2s - 1) \leqslant P_n^{(\alpha,\beta)}(2x - 1)\, P_n^{(\alpha,\beta)}(2y - 1), \tag{2.3}$$

whenever $1 \leqslant r \leqslant x \leqslant y \leqslant s$ *and* $rs = xy$.

3. AN UPPER BOUND FOR $C_n^\lambda(x)\, C_n^\lambda(y)$

THEOREM 6. *If* $\lambda > 0$ *and* $x, y \geqslant 1$ *then*

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} \frac{C_n^\lambda(y)}{C_n^\lambda(1)} \leqslant \frac{C_n^\lambda(xy + (x^2 - 1)^{1/2} (y^2 - 1)^{1/2})}{2C_n^\lambda(1)} + \frac{C_n^\lambda(xy - (x^2 - 1)^{1/2} (y^2 - 1)^{1/2})}{2C_n^\lambda(1)}, \tag{3.1}$$

with equality only when $x = 1$ *or* $y = 1$.

Proof. Rewrite Gegenbauer's formula [2, p. 177]

$$\frac{C_n^\lambda(x)\, C_n^\lambda(y)}{C_n^\lambda(1)} = \frac{\int_0^\pi C_n^\lambda(xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos\theta)(\sin\theta)^{2\lambda-1}\,d\theta}{\int_0^\pi (\sin\theta)^{2\lambda-1}\,d\theta} \tag{3.2}$$

in the form

$$\frac{C_n^\lambda(x)\, C_n^\lambda(y)}{C_n^\lambda(1)} = \frac{\int_0^{\pi/2} C_n^\lambda(xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos\theta)(\sin\theta)^{2\lambda-1}\,d\theta}{2\int_0^{\pi/2} (\sin\theta)^{2\lambda-1}\,d\theta}$$
$$+ \frac{\int_0^{\pi/2} C_n^\lambda(xy + (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos\theta)(\sin\theta)^{2\lambda-1}\,d\theta}{2\int_0^{\pi/2} (\sin\theta)^{2\lambda-1}\,d\theta}, \tag{3.3}$$

$\lambda > 0$. Now use the strict convexity of $C_n^\lambda(t)$ for $t > 1$ (this is clear from [11, (4.7.6)]) to see that

$$C_n^\lambda(xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos\theta) + C_n^\lambda(xy + (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos\theta)$$
$$\leqslant C_n^\lambda(xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}) + C_n^\lambda(xy + (x^2-1)^{1/2}\,(y^2-1)^{1/2}),$$

which, combined with (3.3), gives (3.1).

Remarks. (i). The special case of (3.1) when $x = y$ and $\lambda = 1/2$, so that $C_n^\lambda(x)$ reduces to the Legendre polynomial, was found by Malkov [8]. In this case the inequality also holds for $0 \leqslant x \leqslant 1$, and thus for all real x, since both sides are even functions.

(ii). Setting $x = \cosh\theta$, $y = \cosh\varphi$ in (3.1) and letting $\lambda \to 0$ gives

$$\cosh n\theta \cosh n\varphi \leqslant \tfrac{1}{2}\cosh n(\theta+\varphi) + \tfrac{1}{2}\cosh n(\theta-\varphi); \tag{3.4}$$

and there is equality in (3.4) for all θ, φ. If we let $\lambda \to \infty$ in (0.5), then the first inequality becomes $(xy)^n \leqslant x^n y^n$, in which equality holds for all x, y. Similarly, (2.3) reduces to equality when $\alpha \to \infty$.

(iii). Since $T_n(\cosh\theta) = \cosh n\theta$, the fact that equality holds in (3.4) gives the following simple proof of (0.3):

$$T_n(x)\,T_n(y)$$
$$= \frac{T_n(xy + (x^2-1)^{1/2}\,(y^2-1)^{1/2}) + T_n(xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2})}{2}$$
$$\geqslant T_n\left(\frac{xy + (x^2-1)^{1/2}\,(y^2-1)^{1/2} + xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}}{2}\right)$$
$$= T_n(xy), \qquad x, y \geqslant 1,$$

with equality only when $x = 1$ or $y = 1$, since $T_n(x)$ is a strictly convex function for $x \geqslant 1$ and $xy - (x^2 - 1)^{1/2}(y^2 - 1)^{1/2} \geqslant 1$ when $x, y \geqslant 1$. This convexity argument can be applied to (3.2) to derive the first inequality in (0.5). Application of this argument to an integrated form of Koornwinder's addition formula for Jacobi polynomials [7] leads to the special case

$$\alpha \geqslant \beta \geqslant -\tfrac{1}{2}, \qquad r = 1 \text{ of } (2.3).$$

The first inequality in (0.5) also follows from the case $\beta = -\frac{1}{2}$ of (2.3) by use of a quadratic transformation [11, (4.1.5)] One can give a simple proof of (1.3) for symmetric polynomials by first proving (1.3) for symmetric polynomials of degree 1 and 2, which can easily be done directly, and then forming products of such polynomials.

(iv) The results in [5] can be used to obtain some modifications of our inequalities. For instance, inequality (6) of [5] is equivalent to the fact that if $p_n(x)$ is a polynomial of degree n with only real roots, then

$$\frac{d^2}{dx^2} \log |p_n(x)| + \frac{1}{n}\left(\frac{d}{dx} \log |p_n(x)|\right)^2 \leqslant 0,$$

whenever $p_n(x) \neq 0$.

Note added in proof.

(v) If all the zeros of $p_n(x)$ have real part equal to zero then inequality (1.3) is reversed for all real r, x, y, s with $r \leqslant x \leqslant y \leqslant s$, $rs = xy$, unless $rs < 0$ and $x = 0$ is a root of odd multiplicity, in which case (1.3) holds. This is clearly true for $p_1(x) = x$ and a simple calculation shows that it holds for $p_2(x) = x^2 + a^2$, $a > 0$. The general result follows by multiplication.

(vi) Gegenbauer's addition formula can be used to obtain

$$\frac{C_n^\lambda(2x^2 - 1)}{C_n^\lambda(1)} + 1 = 2\left(\frac{C_n^\lambda(x)}{C_n^\lambda(1)}\right)^2 + 2\sum_{k=1}^{[n/2]} \frac{(2\lambda - 1)_{2k}\,(n + 2\lambda)_{2k}\,(-n)_{2k}}{(\lambda - \frac{1}{2})_{2k}\,(1)_{2k}\,(\lambda + \frac{1}{2})_{2k}\,2^{4k}} \cdot (1 - x^2)^{2k}\left(\frac{C_{n-2k}^{\lambda+2k}(x)}{C_{n-2k}^{\lambda+2k}(1)}\right)^2,$$

so

$$\frac{C_n^\lambda(2x^2 - 1)}{C_n^\lambda(1)} + 1 \geqslant 2\left(\frac{C_n^\lambda(x)}{C_n^\lambda(1)}\right)^2, \qquad \lambda \geqslant 0, \qquad \text{all real } x,$$

and

$$\frac{C_n^\lambda(2x^2 - 1)}{C_n^\lambda(1)} + 1 \leqslant 2\left(\frac{C_n^\lambda(x)}{C_n^\lambda(1)}\right)^2, \qquad -\tfrac{1}{2} < \lambda \leqslant 0, \quad \text{all real } x.$$

This extends Malkov's inequality to ultraspherical polynomials.

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